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A note on the Chevalley-Eilenberg Cohomology for the Galilei and Poincaré Algebras

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Abstract

We construct in a systematic way the complete Chevalley-Eilenberg cohomology at form degree two, three and four for the Galilei and Poincaré groups. The corresponding non-trivial forms belong to certain representations of the spatial rotation (Lorentz) group. In the case of two forms they give all possible central and non-central extensions of the Galilei group (and all non-central extensions of the Poincaré group). The procedure developed in this paper can be applied to any space-time symmetry group.

1 Introduction and Results

Physical theories are characterized by their symmetries. The basic symmetry is that of the underlying space: the transformations that are implied by the homogeneity and isotropy of space. Physical theories respect these kinematical symmetries and exhibit additional dynamical symmetries. Among the possible kinematical groups, which have been classified in [1], the Poincaré and Galilei groups are the better known ones. Here we will study the central and non-central extensions of these two groups, obtaining extended symmetries which can be used to construct new physical theories or to interpret group theoretically certain symmetry properties of known theories.

It is well known that the Galilei group in d+1 space-time dimensions admits a central extension [2], while in the special case of 2+1 dimensions, it admits an exotic, two-fold central extension [3, 4, 5]. This exotic Galilei group has attracted attention in the context and non-commutative geometry and in condensed matter physics [6, 7, 8, 9], see also [10] for further references. In the particular case of $1+1$ dimensions the Galilei algebra has two central extensions and the Poincaré algebra one[1] [11]. The Poincaré group has no central extensions in d+1 ($d > 1$) space-time dimensions; however it has a non-central antisymmetric tensor extension, which has an important physical application: it describes the motion of a relativistic particle in a constant electromagnetic field [15, 16, 17], see also [18].

The extensions of a group are related to a non-trivial Chevalley-Eilenberg (CE) cohomology of the corresponding unextended Lie algebras, for a review see [19]. In this paper we compute in a systematic, almost algorithmic, way the most general CE cohomology groups at form degree two, three and four. As we will see the non-trivial forms belong to different representations of the subgroup of spatial rotations (Lorentz) of the Galilei (Poincaré) group.

In all space-time algebras we can separate the generators into "translations" and "rotations". The "rotations" generators are a subgroup of the automorphism group and constitute a normal subgroup of every extended group. In the case of the Galilei group we consider as "translations" the space and time translations and the galilean boosts, while the "rotations" are the ordinary spatial rotations. For the Poincaré group "translations" are the space-time translations and the "rotations" are the Lorentz transformations.

The first extension, central and non-central, of the unextended Galilei algebra is obtained by calculating all possible non-trivial closed 2-forms of the subgroup of "translations". The forms are closed with respect the exterior differential operator d. These forms can be grouped into different representations of the rotation group: a vector, an antisymmetric tensor and a symmetric tensor. (In the case of Poincaré there is only an antisymmetric tensor representation.) The trace of the symmetric tensor corresponds to the well-known central extension of the Galilei group [2]. In the case of 2+1 dimensions the antisymmetric tensor becomes a pseudoscalar representation which gives rise to the

[1]These $1+1$ extended algebras have been used to study several problems of gravity and Moyal quantization, see for example [12, 13, 14].

second central extension.

The complete extended algebra is constructed from the original algebra and the extensions by incorporating their transformation properties under rotations, which is equivalent to replacing the exterior differential operator d by the corresponding "covariant" operator $d + M\wedge$, M being the zero-curvature connection associated to the "rotations": $dM + M \wedge M = 0$.

As we will see, once we have an extended algebra, we can further extend it by applying the previous procedure to the new set of translations that now includes the extended generators. In this way we obtain new extensions with some generators belonging to higher dimensional representations of the corresponding "rotations" group. The physical significance of these higher dimensional extensions should be clarified. At this level the generators of the first and second level extensions couple non-trivially to each other. This procedure for the case of the Galilei and Poincaré groups does not end.

We have also studied the first level CE cohomology at degrees three and four. As in the case of 2-forms, the non-trivial 3- or 4-forms obtained are grouped into representations of the "rotations" group. The non-invariant 2-form "potentials" associated to these closed 3-forms could be used to construct non-relativistic string theories whose Lagrangian will be invariant up to a total derivative under the Galilei (Poincaré) group.

For the Galilei group these string theories will be different from the non-relativistic string theory introduced in [20], [21], [22] because the invariance group of this string theory is the Galilean string group which is obtained by a stringy contraction of the Poincaré group [23] and not by the ordinary contraction that leads to the ordinary Galilei algebra. In the case of the Poincaré group the forms can be used to construct Wess-Zumino (WZ) terms for relativistic string theories coupled to external backgrounds. Finally, we note that the non-trivial closed 4-forms obtained here could be used to construct WZ actions of non-relativistic and relativistic branes.

The procedure to compute CE cohomolgy groups employed in this note is described in the Appendix and can be applied to any space-time symmetry group. The results are given for the case of 3+1 dimensions, but this procedure, mutatis mutandis, can be used in any number of space dimensions. The application to super space-time groups with odd generators will be done in a forthcoming paper.

The organization of the paper is as follows. In section 2 we give our notation and conventions and study the first level extension of the Galilei group. In section 3 we analyze the second level extension. In section 4 we give the results of the first level CE cohomology at form-degree three and four, while in section 5 we give the corresponding results for the Poincaré group. Finally, in an Appendix, we describe the "algorithm" used for obtaining these extensions. The calculations make use of the first author's Mathematica package EDC (Exterior Differential Calculus) [24].

2 Galilei group in 3+1 dimensions

The generators of the unextended Galilei algebra are the hamiltonian, H, the spatial translations P_a, the boots K_a, and the rotations M_{ab}. The algebra is given by[2]

$$\begin{aligned}
[M_{ab},\ M_{cd}] &= -i\ \eta_{b[c}\ M_{ad]} + i\ \eta_{a[c}\ M_{bd]}, \\
[K_a,\ M_{cd}] &= -i\ \eta_{a[c}\ K_{d]}, \\
[P_a,\ M_{bc}] &= -i\ \eta_{a[b}\ P_{c]}, \\
[H,\ M_{ab}] &= 0, \\
[H,\ K_a] &= +i\ P_a, \\
[H,\ P_a] &= 0, \\
[K_a,\ K_b] &= 0, \\
[P_a,\ K_b] &= 0, \\
[P_a,\ P_b] &= 0,
\end{aligned} \tag{2.1}$$

where η_{ab} is the euclidean metric of 3-dimensional space, $a, b = 1,\ 2,\ 3$.

In order to construct the extensions it is very useful to introduce the left invariant Maurer-Cartan (MC) form, defined by

$$\Omega = -ig^{-1}dg, \tag{2.2}$$

where g represents a general element of the Galilei group. The MC form satisfies the Maurer-Cartan equation

$$d\Omega + i\ \Omega \wedge \Omega = 0. \tag{2.3}$$

In components the MC 1-form is written, for a generic Lie algebra, as

$$\Omega = X_A\ \mathcal{X}^{\mathcal{A}}, \tag{2.4}$$

where X_A are the generators of the Lie algebra satisfying

$$[X_B,\ X_C] = i\ f^A{}_{BC}\ X_A \tag{2.5}$$

and $\mathcal{X}^{\mathcal{A}}$ are corresponding 1-forms. (Throughout this paper we use the same capital letters in plain and calligraphic font to denote generators and associated 1-forms). The MC equation (2.3) implies that the 1-forms $\mathcal{X}^{\mathcal{A}}$ satisfy

$$d\mathcal{X}^A = \frac{1}{2} f^A{}_{BC} \mathcal{X}^B \wedge \mathcal{X}^C. \tag{2.6}$$

For the Galilei case, the MC 1-form (2.4) becomes

$$\Omega = H\ \mathcal{H} + P_a\ \mathcal{P}^a + K_a\ \mathcal{K}^a + \frac{1}{2} M_{ab}\ \mathcal{M}^{ab}, \tag{2.7}$$

[2] Although the algebra is real and the imaginary units can be made to disappear by replacing all generators G by iG', we prefer to leave the i's in the equations because then we can interpret the generators as Hermitian operators.

while the MC equation (2.3) in components is

$$\begin{aligned} d\mathcal{H} &= 0, \\ d\mathcal{P}^a &+ \mathcal{P}^c \wedge \mathcal{M}_c{}^a - \mathcal{H} \wedge \mathcal{K}^a = 0, \\ d\mathcal{K}^a &+ \mathcal{K}^c \wedge \mathcal{M}_c{}^a = 0, \\ d\mathcal{M}^{ab} &+ \mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b = 0. \end{aligned} \tag{2.8}$$

In order to start our procedure for the cohomological analysis we need to define the "translations" generators, which we take to be the space and time translations and the boosts, i.e., H, P_a, K_a. The MC equations for these generators are obtained by putting $\mathcal{M}^{ab} \to 0$ in (2.8).

The most general closed invariant 2-form which cannot be written as the differential of an invariant 1-form is then found to be

$$\Omega_2 = f_a\, \mathcal{H} \wedge \mathcal{P}^a + f_{[ab]}\, \mathcal{K}^a \wedge \mathcal{K}^b + f_{(ab)}\, \mathcal{K}^a \wedge \mathcal{P}^b \tag{2.9}$$

where the constant parameters are a vector f_a, a second rank antisymmetric tensor $f_{[ab]}$ and a second rank symmetric tensor $f_{(ab)}$[3]. Therefore we find that the non-trivial 2-forms belong to a vector, a symmetric and an antisymmetric tensor representation of the rotation group. The 1-form "potentials" associated to these 2-forms are denoted by

$$\mathcal{Z}^a,\ \mathcal{Z}^{[ab]},\ \mathcal{Z}^{(ab)} \tag{2.10}$$

and satisfy he MC equations

$$\begin{aligned} d\mathcal{Z}^a &= \mathcal{H} \wedge \mathcal{P}^a \\ d\mathcal{Z}^{[ab]} &= \mathcal{K}^a \wedge \mathcal{K}^b \\ d\mathcal{Z}^{(ab)} &= \mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a. \end{aligned} \tag{2.11}$$

From these expressions we can obtain the algebra of the corresponding generators, denoted by

$$Z_a,\ Z_{[ab]},\ Z_{(ab)}. \tag{2.12}$$

We find

$$\begin{aligned} [K_a,\ K_b] &= +i\ Z_{[ab]} \\ [P_a,\ K_b] &= -i\ Z_{(ab)}, \\ [H,\ P_a] &= +i\ Z_a. \end{aligned} \tag{2.13}$$

The extension Z_a has been used to describe the motion of a particle in a constant field, $Z_{[ab]}$ to study the motion of a particle in a non-commutative space, while $Z_{(ab)}$ appears in the description of a particle with mass anisotropy. The ordinary central extension of the Galilei group corresponds to the trace part of $Z_{(ab)}$. In 2 +1 dimensions, the

[3]The calculation proceeds in several distinct steps, as outlined in the Appendix.

second (exotic) central extension corresponds to the single independent component of the antisymmetric tensor $Z_{[ab]}$, while in 1+1 dimensions the single-component vector extension Z_a gives the extra central charge of Galilei [11].

With the rotations included, the extended set of MC 1-forms satisfies the equations

$$\begin{aligned} d\mathcal{H} &= 0, \\ d\mathcal{P}^a &= -\mathcal{P}^c \wedge \mathcal{M}_c{}^a + \mathcal{H} \wedge \mathcal{K}^a, \\ d\mathcal{K}^a &= -\mathcal{K}^c \wedge \mathcal{M}_c{}^a, \\ d\mathcal{M}^{ab} &= -\mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b, \\ d\mathcal{Z}^a &= -\mathcal{Z}^c \wedge \mathcal{M}_c{}^a + \mathcal{H} \wedge \mathcal{P}^a \\ d\mathcal{Z}^{[ab]} &= -\mathcal{Z}^{[ac]} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Z}^{[cb]} + \mathcal{K}^a \wedge \mathcal{K}^b, \\ d\mathcal{Z}^{(ab)} &= -\mathcal{Z}^{(ac)} \wedge \mathcal{M}_c{}^b - \mathcal{M}^a{}_c \wedge \mathcal{Z}^{(cb)} + \mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a. \end{aligned} \tag{2.14}$$

2.1 Explicit parametrization

Here we will obtain an explicit parametrization for all MC 1-forms in terms of differentials of functions, so that equations (2.14) are satisfied. We first obtain expressions for the unextended MC 1-forms without rotations. Locally, we can parametrize a general element of the unextended group by

$$g = e^{iHx^0} e^{iP_a x^a} e^{iK_a v^a}. \tag{2.15}$$

The MC 1-form (2.2) can be computed directly from its definition using the Baker-Campbell-Hausdorff formula. The components of the MC 1-form (2.4) can be computed explicitly using Mathematica and EDC [24], if we substitute a matrix representation for the generators, for example the adjoint representation. The result is

$$\begin{aligned} \mathcal{H} &= dx^0, \\ \mathcal{P}^a &= dx^a - v^a dx^0, \\ \mathcal{K}^a &= dv^a. \end{aligned} \tag{2.16}$$

For the extended group, instead of following the same procedure starting with the general element

$$g = e^{iHx^0} e^{iP_a x^a} e^{iK_a v^a} e^{iZ_a k^a} e^{iZ_{(ab)} k^{(ab)}} e^{iZ_{[ab]} k^{[ab]}}, \tag{2.17}$$

it is much easier to use the known parametrization for the 1-forms $\mathcal{H}$, $\mathcal{P}^a$, $\mathcal{K}^a$ on the right hand side of (2.11) and integrate. The result is that the new MC 1-forms, modulo exact 1-forms, can be written

$$\begin{aligned} \mathcal{Z}^a &= dk^a - x^a\, dx^0, \\ \mathcal{Z}^{[ab]} &= dk^{[ab]} + \frac{1}{2}(v^a\, dv^b - v^b\, dv^a), \\ \mathcal{Z}^{(ab)} &= dk^{(ab)} + v^a\, d(x^b - v^b\, x^0) + v^b\, d(x^a - v^a\, x^0), \end{aligned} \tag{2.18}$$

where k^a, $k^{[ab]}$, $k^{(ab)}$ are new functions – the group parameters associated to the new generators in (2.17). These 1-forms can be used to construct non-relativistic particle

Lagrangians. The study of these Lagrangians and their physical implications will be considered elsewhere. Here let us notice that the familiar free non-relativistic massive particle is obtained by taking the trace of $\mathcal{Z}^{(ab)}$ and coincides with the one obtained with the method of non-linear realizations [25], see for example [26].

If we want to have the explicit expressions of the MC 1-forms when we include rotations, the right hand side of all vector and tensor expressions given above must be multiplied by an appropriate rotation matrix for each index:

$$\mathcal{P}^a = {U^{-1}}^a{}_b\,(dx^b + \cdots), \qquad \mathcal{Z}^{ab} = {U^{-1}}^a{}_p\,{U^{-1}}^b{}_q\,(dk^{pq} + \cdots), \tag{2.19}$$

where the rotation MC 1-forms have the explicit representation $\mathcal{M}^a{}_b = {U^{-1}}^a{}_c\,d[U^c{}_b]$.

3 Second Level Extensions

One can obtain further extensions of the Galilei group corresponding to non-trivial 2-forms in higher dimensional representations of the rotation group. In order to find them we follow the same procedure as in the last section, this time taking as "translations" the nineteen 1-forms

$$\mathcal{H},\ \mathcal{P}^a,\ \mathcal{K}^a,\ \mathcal{Z}^a,\ \mathcal{Z}^{[ab]},\ \mathcal{Z}^{(ab)}. \tag{3.1}$$

We obtain 33 new non-trivial closed 2-forms which are the components of the following tensors:

$$\begin{array}{c} \mathcal{H}\wedge\mathcal{Z}^a, \\ \mathcal{H}\wedge\mathcal{Z}^{(ab)} + \mathcal{K}^a\wedge\mathcal{Z}^b + \mathcal{K}^b\wedge\mathcal{Z}^a, \\ \mathcal{P}^a\wedge\mathcal{P}^b + \mathcal{K}^a\wedge\mathcal{Z}^b - \mathcal{K}^b\wedge\mathcal{Z}^a, \\ \mathcal{H}\wedge\mathcal{Z}^{[ab]} + \mathcal{K}^a\wedge\mathcal{P}^b - \mathcal{K}^b\wedge\mathcal{P}^a, \\ \mathcal{K}^a\wedge\mathcal{Z}^{(bc)} + \mathcal{K}^b\wedge\mathcal{Z}^{(ca)} + \mathcal{K}^c\wedge\mathcal{Z}^{(ab)}, \\ 2\ \mathcal{Z}^{[ab]}\wedge\mathcal{K}^c - \mathcal{Z}^{[bc]}\wedge\mathcal{K}^a - \mathcal{Z}^{[ca]}\wedge\mathcal{K}^b, \end{array} \tag{3.2}$$

the latter being a tensor antisymmetric in the first two indices (ab) whose totally antisymmetric part $\epsilon_{abc}\ \mathcal{Z}^{[ab]}\wedge\mathcal{K}^c$ vanishes.

If we use the kernel symbol $\mathcal{Y}$ for the 1-form tensor "potentials" associated to these 2-forms and include the effect of rotations, we find that they satisfy the equations

$$\begin{array}{rcl} d\mathcal{Y}^a &=& -\mathcal{Y}^c\wedge\mathcal{M}_c{}^a + \mathcal{H}\wedge\mathcal{Z}^a, \\ d\mathcal{Y}^{(ab)} &=& -\mathcal{Y}^{(ac)}\wedge\mathcal{M}_c{}^b - \mathcal{M}^a{}_c\wedge\mathcal{Y}^{(cb)} + \mathcal{H}\wedge\mathcal{Z}^{(ab)} + \mathcal{K}^a\wedge\mathcal{Z}^b + \mathcal{K}^b\wedge\mathcal{Z}^a, \\ d{\mathcal{Y}_1}^{[ab]} &=& -{\mathcal{Y}_1}^{[ac]}\wedge\mathcal{M}_c{}^b - \mathcal{M}^a{}_c\wedge{\mathcal{Y}_1}^{[cb]} + \mathcal{P}^a\wedge\mathcal{P}^b + \mathcal{K}^a\wedge\mathcal{Z}^b - \mathcal{K}^b\wedge\mathcal{Z}^a, \\ d{\mathcal{Y}_2}^{[ab]} &=& -{\mathcal{Y}_2}^{[ac]}\wedge\mathcal{M}_c{}^b - \mathcal{M}^a{}_c\wedge{\mathcal{Y}_2}^{[cb]} + \mathcal{H}\wedge\mathcal{Z}^{[ab]} + \mathcal{K}^a\wedge\mathcal{P}^b - \mathcal{K}^b\wedge\mathcal{P}^a, \\ d\mathcal{Y}^{(abc)} &=& -\mathcal{Y}^{(abs)}\wedge\mathcal{M}_s{}^c - \mathcal{Y}^{(bcs)}\wedge\mathcal{M}_s{}^a - \mathcal{Y}^{(cas)}\wedge\mathcal{M}_s{}^b \\ && +\mathcal{K}^a\wedge\mathcal{Z}^{(bc)} + \mathcal{K}^b\wedge\mathcal{Z}^{(ca)} + \mathcal{K}^c\wedge\mathcal{Z}^{(ab)}, \\ d{\mathcal{Y}_3}^{[ab]c} &=& -\mathcal{M}^a{}_s\wedge{\mathcal{Y}_3}^{[sb]c} - \mathcal{M}^b{}_s\wedge{\mathcal{Y}_3}^{[as]c} - \mathcal{M}^c{}_s\wedge{\mathcal{Y}_3}^{[ab]s} \\ && +2\ \mathcal{Z}^{[ab]}\wedge\mathcal{K}^c - \mathcal{Z}^{[bc]}\wedge\mathcal{K}^a - \mathcal{Z}^{[ca]}\wedge\mathcal{K}^b, \end{array} \tag{3.3}$$

where the symmetries of the $\mathcal{Y}$ tensors are as indicated by round or square brackets, while the totally antisymmetric part of ${\mathcal{Y}_3}^{[ab]c}$ vanishes. The corresponding new algebra generators will be written

$$Y_a,\ Y_{(ab)},\ {Y^1}_{[ab]},\ {Y^2}_{[ab]},\ Y_{(abc)},\ {Y^3}_{[ab]c}. \tag{3.4}$$

We observe that, at this level the generators of the extensions couple non-trivially to each other, for example

$$\Big[K_a,\ Z_{(bc)}\Big] = i\ Y_{(abc)}, \tag{3.5}$$

while the P_a no longer commute:

$$[P_a,\ P_b] = i\ {Y^1}_{[ab]}. \tag{3.6}$$

The generator ${Y^1}_{[ab]}$ can be used to describe the motion of non-relativistic particles in a constant magnetic background.

We could complete the algebra by introducing the transformations of the new generators under rotations. For the case of the Galilei group this extension procedure does not end. The physical interpretation of these new Y terms remains to be understood.

4 CE cohomology at degrees three and four

In this section we study the most general non-trivial closed 3-forms and 4-forms. Our procedure gives 18 closed non-trivial 3-forms, which, as in the case of 2-forms studied above, can be grouped under different representations of the rotation group, resulting in:

- a symmetric tensor $\mathcal{H} \wedge (\mathcal{K}^a \wedge \mathcal{P}^b + \mathcal{K}^b \wedge \mathcal{P}^a)$
- an antisymmetric tensor $\mathcal{H} \wedge \mathcal{P}^a \wedge \mathcal{P}^b$
- a pseudoscalar $\epsilon_{abc}\ \mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{K}^c$
- and the following 3rd rank tensor, antisymmetric in its first 2 indices $[ab]$, whose totally antisymmetric part vanishes

$$2\ \mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{P}^c - \mathcal{K}^b \wedge \mathcal{K}^c \wedge \mathcal{P}^a - \mathcal{K}^c \wedge \mathcal{K}^a \wedge \mathcal{P}^b. \tag{4.1}$$

These closed 3-forms imply the existence of corresponding 2-form tensor potentials[4]. In terms of the explicit parametrization of group elements introduced in (2.16), (2.18), we can write these 2-form potentials (modulo exact of 2-forms) as

$$\begin{gathered} -\,(v^a\ dx^0 \wedge dx^b + v^b\ dx^0 \wedge dx^a), \\ -\frac{1}{2}(x^a\ dx^0 \wedge dx^b - x^b\ dx^0 \wedge dx^a), \\ \epsilon_{abc}\ v^a\ dv^b \wedge\ dv^c, \\ vvx(a,b,c) - \frac{1}{2} vvx(b,c,a) - \frac{1}{2} vvx(c,a,b), \end{gathered} \tag{4.2}$$

[4]Closed differential systems of tensor valued differential forms of any order are known as "Free Differential Algebras" (FDAs), see [27] [28].

where $vvx(a,b,c)$ stands for $(v^a\, dv^b - v^b\, dv^a) \wedge (dx^c - v^c\, dx^0)$.

These 2-forms could be used to construct non-relativistic string theories whose Lagrangian is invariant up to an exact form under the Galilei group. As mentioned in the Introduction, these string theories will be different from the non-relativistic string theory introduced in [20], [21], [22].

Finally, applying our procedure at degree four we obtain 19 non-trivial closed 4-forms which are the components of the following tensors:

$$\begin{gathered} \mathcal{H} \wedge \mathcal{K}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c, \\ \mathcal{H} \wedge \epsilon_{abc}\, \mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c, \\ \mathcal{P}^d \wedge \epsilon_{abc}\, \mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{K}^c, \\ \mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{K}^c \wedge \mathcal{K}^d + \mathcal{K}^a \wedge \mathcal{K}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d. \end{gathered} \tag{4.3}$$

The corresponding non-trivial forms could be used for constructing WZ terms of non-relativistic branes. In higher dimensions, the terms containing ϵ_{abc} would give totally antisymmetric third rank tensors.

5 Poincaré Extensions

It is known that the Poincaré group admits no central extensions in $d > 1$ dimensions. However it has an antisymmetric tensor non-central extension [15, 16, 17, 18]. Here we apply our procedure to the Poincaré group and obtain all non-central extensions. We will use the same notation as in the previous sections, but the tensor indices will now take the values (0, 1, 2, 3) while η_{ab} will denote the Minkowski metric. The generators now satisfy the equations

$$\begin{aligned} [M_{ab},\ M_{cd}] &= -i\ \eta_{b[c}\ M_{ad]} + i\ \eta_{a[c}\ M_{bd]}, \\ [P_a,\ M_{bc}] &= -i\ \eta_{a[b}\ P_{c]} \end{aligned} \tag{5.1}$$

and the corresponding MC 1-forms satisfy

$$\begin{aligned} d\mathcal{P}^a &= -\mathcal{M}^a{}_c \wedge \mathcal{P}^c, \\ d\mathcal{M}^{ab} &= -\mathcal{M}^{ac} \wedge \mathcal{M}_c{}^b. \end{aligned} \tag{5.2}$$

Freezing the Lorentz freedom ($\mathcal{M}^a{}_b \to 0$), the first-level extension results in the antisymmetric tensor 2-form $\mathcal{P}^a \wedge \mathcal{P}^b$. Using the same notation for the corresponding potential 1-form ($\mathcal{Z}^{[ab]}$) and reintroducing, as before, the Lorentz freedom, we arrive at

$$d\mathcal{Z}^{[ab]} = -\mathcal{M}^a{}_c \wedge \mathcal{Z}^{[cb]} - \mathcal{M}^b{}_c \wedge \mathcal{Z}^{[ac]} + \mathcal{P}^a \wedge \mathcal{P}^b, \tag{5.3}$$

which implies that the generators of translations no longer commute:

$$[P_a,\ P_b] = i\ Z_{[ab]}. \tag{5.4}$$

As there are no other extensions, the well known result that there are no central extensions for the Poincaré group follows. This antisymmetric generator $Z_{[ab]}$ can be used to describe the motion of a relativistic particle in a constant electromagnetic field.

The calculation of the second level extensions results in 20 closed non-trivial 2-forms which can be written as the components the tensor[5]

$$2\ \mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \tag{5.5}$$

Again, introducing the second-level potential 1-form $\mathcal{Y}^{a[bc]}$ with the same symmetries as the above 2-form tensor and unfreezing the Lorentz freedom, we find

$$\begin{aligned} d\mathcal{Y}^{a[bc]} &= -\mathcal{M}^a{}_s \wedge \mathcal{Y}^{s[bc]} - \mathcal{M}^b{}_s \wedge \mathcal{Y}^{a[sc]} - \mathcal{M}^c{}_s \wedge \mathcal{Y}^{a[bs]} \\ &\quad + 2\ \mathcal{P}^a \wedge \mathcal{Z}^{[bc]} - \mathcal{P}^b \wedge \mathcal{Z}^{[ca]} - \mathcal{P}^c \wedge \mathcal{Z}^{[ab]}. \end{aligned} \tag{5.6}$$

As in the Galilei case, the generators of first and second level extensions now couple non-trivially to each other

$$\left[P_a,\ Z_{[bc]}\right] = 2\ i\ Y_{a[bc]} - i\ Y_{b[ca]} - i\ Y_{c[ab]}. \tag{5.7}$$

Finally, the CE cohomology at higher degrees gives simple non-trivial results. For degree 3 there are 4 closed 3-forms which are the components of the pseudovector

$$\epsilon_{abcd}\ \mathcal{P}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d, \tag{5.8}$$

while at degree 4 there is a single pseudoscalar 4-form: $\epsilon_{abcd}\ \mathcal{P}^a \wedge \mathcal{P}^b \wedge \mathcal{P}^c \wedge \mathcal{P}^d$.

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Appendix

The calculations in this note are done using the Mathematica package EDC (Exterior Differential Calculus) [24] and are contained in the Mathematica notebooks GalileiSequentialExtensions.nb and PoincareSequentialExtensions.nb, which are available from the

[5] This tensor is antisymmetric in $[bc]$ and its totally antisymmetric part vanishes. This leads to 4 identities, $\epsilon_{abcd}\mathcal{P}^b \wedge \mathcal{Z}^{[cd]} = 0$, leaving 4x6-4=20 independent components.

authors. The calculations can be broken down into the following five distinct steps (results given for the Galilei case):

Step 1: Set $\mathcal{M}^{ab} = 0$ and construct all possible 2-forms from the set of "translations" $\{\mathcal{H},\ \mathcal{K}^a,\ \mathcal{P}^a\}$, obtaining a set of twenty-one 2-forms. Find those combinations of these 2-forms that are closed and not trivial (cannot be written as d[1-form]). This leads to the following twelve 2-forms

$$\{\mathcal{H}\wedge\mathcal{P}^1,\ \mathcal{H}\wedge\mathcal{P}^2,\ \mathcal{H}\wedge\mathcal{P}^3,\ \mathcal{K}^1\wedge\mathcal{K}^2,\ \mathcal{K}^1\wedge\mathcal{K}^3,\ \mathcal{K}^2\wedge\mathcal{K}^3,\ \mathcal{K}^1\wedge\mathcal{P}^1,\ \mathcal{K}^2\wedge\mathcal{P}^2,\\ \mathcal{K}^1\wedge\mathcal{P}^2+\mathcal{K}^2\wedge\mathcal{P}^1,\ \mathcal{K}^1\wedge\mathcal{P}^3+\mathcal{K}^3\wedge\mathcal{P}^1,\ \ \mathcal{K}^2\wedge\mathcal{P}^3+\mathcal{K}^3\wedge\mathcal{P}^2,\ \mathcal{K}^3\wedge\mathcal{P}^3\}$$

Step 2: Group these twelve 2-forms as the components of a vector $\mathcal{H}\wedge\mathcal{P}^a$, an antisymmetric tensor $\mathcal{K}^a\wedge\mathcal{K}^b$, and a symmetric tensor $\mathcal{K}^a\wedge\mathcal{P}^b+\mathcal{K}^b\wedge\mathcal{P}^a$. This step is the only non-algorithmic part of the procedure and can be non-trivial.

Step 3: Introduce the rotations and verify that these 2-forms remain "closed" if we replace d by the "covariant" exterior derivative $d+\mathcal{M}\wedge$.

Step 4: Now, if a tensor valued 2-form $\mathcal{F}^a$ satisfies the equation[6] $d\mathcal{F}^a+\mathcal{M}^a{}_b\wedge\mathcal{F}^b=0$, then there exists a corresponding tensor valued "potential" 1-form $\mathcal{Z}^a$ satisfying

$$d\mathcal{Z}^a+\mathcal{M}^a{}_b\wedge\mathcal{Z}^b=\mathcal{F}^a. \tag{5.9}$$

Introduce notation for the components of these 1-form potentials and write the component form of these equations. Verify consistency.

Step 5: Combine with the original algebra to obtain the extended algebra of MC 1-forms given in (2.14). Translate this algebra to the algebra of commutators of the corresponding generators.

The entire procedure of 5 steps is then repeated for the extended set of 2-forms (second level extension). The space of 2-forms now consists of 171 elements, of which 33 are found to be closed, non-trivial and not included in the first extension. They are the components of the tensors given in section 3.

Finally, the first two steps for the first-level 3-form and 4-form cohomology are also given. We find 18 closed non-trivial 3-forms and 19 closed non-trivial 4-forms which can be grouped as the components of the tensors given in section 4. Steps 3 and 4 (requiring now 2- or 3-form potentials) are straightforward.

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